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Darboux theorem and equivariant Morse lemma

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Abstract

Using supersymmetry differential forms can be studied by methods of the theory of normal forms of smooth functions. Differential forms on M can be considered as functions on a supermanifold $\widehat{M}$ and the De Rham differential on M becomes a vector field on $\widehat{M}$. This vector field generates an action of the supergroup $\mathbb{R}^{0|1}$ on $\widehat{M}$. We prove the equivariant Morse lemma for this action and show that it implies both the ordinary Morse lemma and the Darboux theorem on M.

Keywords: Supersymmetry; Differential forms; Morse lemma; Darboux theorem
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0. Introduction

0.1. Singularities of functions and differential forms

Let us consider the basic facts about local behavior of the two types of objects on manifolds: smooth functions and closed differential forms. The two cases look surprisingly similar.

Let M be a C^∞-manifold with a point $P \in M$, smooth vanishing at P function f, and closed differential form α.

- If f is non-singular at P (that is, the linear part of its Taylor expansion at P is non-zero), then there exists a local coordinate system $x_1, \dots, x_n$ at P, such that $f(x_1, \dots, x_n) = x_1$ (*Implicit function theorem*).
- If α is a closed 1-form that does not vanish at P, then there exists a local coordinate system $x_1, \dots, x_n$ at P in which $\alpha = dx_1$.

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- If f is singular at P but its second differential $f_0 \in S^2(T_P^*M)$ (the quadratic part of the Taylor expansion) is a non-degenerate quadratic form on $T_P M$, then there exists a local coordinate system in which f coincides with its quadratic part f_0 (*Morse lemma*).
- If α is a closed 2-form, such that its restriction to $T_P M$ is a non-degenerate skew form $\alpha_0 \in \Lambda^2(T_P^*M)$, then there exists a local coordinate system in which α has constant coefficients and, therefore, coincides with $\alpha_0 = \alpha(0)$ (*Darboux theorem*).

The first two theorems do not only look similar, they are essentially equivalent. The first theorem is a particular case of the implicit function theorem and the second follows from it because any closed 1-form is (locally) a differential of a function vanishing at P.

The other two theorems (the Morse lemma and the Darboux theorem), however, do not follow from each other. They are considered as starting points of two different branches of Geometry: Morse theory and symplectic geometry.

The aim of this paper is to show that this analogy is not superficial, and that the Darboux theorem and the Morse lemma are intimately related. They both are particular cases of a more general theorem—the equivariant Morse lemma for supermanifolds which is proved here.

0.2. Supersymmetry and supermanifolds

The idea of supersymmetry was developed originally by physicists in their attempt to construct a theory unifying all known kinds of interactions. Supersymmetry in physics means simultaneous consideration of particles with different types of statistics (bosons and fermions). Even though the theory of Grand Unification is still far from being completed, the supersymmetry approach in physics has already proved to be very useful.

The branch of mathematics that provides ground for these ideas is called supermanifold theory. Many concepts and results of classical geometry, analysis and algebra have been successfully generalized for the super case and have been used to justify physical constructions and statements (cf. [5,10]). The idea of supersymmetry turned out to be fruitful even in very traditional parts of mathematics. It has brought new proofs, new results and new insights in several classical fields. Examples include the Atiyah–Singer index theorem [7], equivariant characteristic classes [11], the Alexander knot polynomial [15], Morse theory [21], and Weyl formula for representations of simple Lie groups [1].

We are not going to discuss these applications in this paper. Our goal here is to emphasize the unifying role that supersymmetry plays in mathematics. Roughly speaking, supersymmetry in mathematics makes "equal" commuting and anticommuting objects. From this point of view, objects which look very different become closely related. Here are a few examples: polynomials and skew forms on a vector space; the orthogonal and the symplectic groups; differential operators and elements of the Clifford algebra; finite- and infinite-dimensional Lie algebras; integration and differentiation; smooth functions and differential forms.

We will elaborate only on the last example and will explain how the Morse lemma and the Darboux theorem may be treated as two particular cases of one theorem.

0.3. Darboux theorem as Morse lemma. Plan of the proof

The main idea is very simple. Differential forms on an n-dimensional manifold M can be considered as functions on an $n|n$-dimensional supermanifold $\widehat{M}$. That is, $C^\infty(\widehat{M}) = \Omega^*_M$, the algebra of differential forms on M. For example, a 2-form $\omega = \sum f_{ij}(x)\, dx_i \wedge dx_j$ becomes the function $F(x,\xi) = \sum f_{ij}(x)\xi_i\xi_j$, where ξ_i is the odd coordinate on $\widehat{M}$ corresponding to dx_i.

All results of classical analysis can be generalized to the case of supermanifolds. In particular, we can use the super analogs of the implicit function theorem and the Morse lemma and find a diffeomorphism of $\widehat{M}$ reducing ω to its quadratic part $\omega_0 = \sum f_{ij}(0)\, dx_i \wedge dx_j$.

This, however, does not prove the Darboux theorem, because we need a diffeomorphism coming from M; but not every diffeomorphism of the supermanifold $\widehat{M}$ is induced by a diffeomorphism of M.

To get rid of the extra freedom that exists on $\widehat{M}$, we can use an additional structure on $\widehat{M}$—a vector field $D = \sum_i \xi_i(\partial/\partial x_i)$ that corresponds to the De Rham differential in Ω^*_M. The vector field D is odd, that is, it changes parity. Unlike the case of even vector fields, not every odd vector field X on a supermanifold can be integrated. The reason is that for odd vector fields the (super)commutator $[X, X] = 2X^2$ is not necessarily zero. For the De Rham vector field we have $[D, D] = 2D^2 = 0$, which means that the vector field D can be integrated to a flow. This flow gives a canonical action of the supergroup $G = \mathbb{R}^{0|1}$ on the supermanifold $\widehat{M}$.

The condition that the differential form ω is closed is equivalent to the invariantness of the corresponding function on $\widehat{M}$ with respect to the De Rham flow. The equivariant Morse lemma (which is proved in Section 3) asserts that the form ω can be reduced to its quadratic part ω_0 by a G-equivariant coordinate transformation F (i.e. by a diffeomorphism of $\widehat{M}$ which commutes with this flow). Using the description of G-equivariant diffeomorphisms given in Section 2, we show that the diffeomorphism $\bar{F}$ of M, obtained from F by forgetting the odd coordinates, takes the differential form ω to its quadratic part ω_0.

In the same spirit, the super version of the Darboux theorem [8,17] and the super Morse lemma [16] follow from the equivariant Morse lemma for the De Rham action on $\widehat{M}$, where M is now a supermanifold. This approach also allows one to study non-homogeneous differential forms, that is forms $\omega = \sum \omega_i$, where $\omega_i \in \Omega^i_M$. Such forms appear in Quillen's approach to the Chern character [11] and the Atiyah–Jeffrey theory [3].

0.4. The structure of the paper

One of the goals of this paper is to demonstrate usefulness of the supermanifold theory, therefore we do not assume familiarity with this field. In Section 1 we present the definitions and results which are used later.

In Section 2 we study the supermanifold $\widehat{M}$, the algebra of functions on which is the algebra of differential forms Ω^*_M on a manifold M. This supermanifold was considered previously by different authors. Very interesting applications of this construction were found by Bernstein–Leites [6] and Witten [21]. The supermanifold $\widehat{M}$ may be considered as an odd analog of the loop space since $\widehat{M} \simeq \mathrm{Map}(\mathbb{R}^{0|1}, M)$, and $\mathbb{R}^{0|1}$ is the only $0|1$-dimensional supergroup—the odd analog of the circle group. And again, the peculiarities of the super case make the odd loop space different from its even counterpart in at least one essential aspect: the super loop space is finite-dimensional. Even loop spaces are equipped with an S^1-action. On $\widehat{M}$ there is a canonical action of the supergroup $\mathbb{R}^{0|1}$. The generator of this action (an odd vector field D) is the De Rham differential in Ω^*_M. We describe D-equivariant diffeomorphisms of $\widehat{M}$ in Section 2. Then in Section 3, we use this description to prove the equivariant Morse lemma for the De Rham action and show that it implies the Darboux theorem on M. For odd-dimensional manifold M our theorem gives the first classification result on degenerate 2-forms (also due to Darboux). More complicated singularities of closed forms considered by Martinet [12] and Roussarie [14] can also be studied by this method.

1. Preliminaries on supermanifolds

In this section we summarize definitions and facts from the theory of supermanifolds which will be used in the paper. Details can be found in [5,8–10]. In what follows we refer to original publications only if the corresponding results are not covered in these references.

1.1. Superspaces and supermanifolds

In the supermanifold theory everything is supposed to be $\mathbb{Z}_2$-graded. Grading (or *parity*) of an object a is denoted by $\bar{a}$. Elements of parity $\bar{0}$ are called *even* and elements of parity $\bar{1}$ are called *odd*. Even and odd elements are called *homogeneous*. All standard algebraic notions can be generalized to the $\mathbb{Z}_2$-graded case by introducing signs in proper places. The following main principle helps to have the signs right.

When something of parity p moves behind something of parity q, the sign $(-1)^{pq}$ *appears.*

The $(p|q)$-dimensional *superspace* $\mathbb{R}^{p|q}$ is a pair $(\mathbb{R}^p, C^\infty(\mathbb{R}^{p|q}))$, where $C^\infty(\mathbb{R}^{p|q}) = C^\infty(\mathbb{R}^p) \otimes \Lambda^*(\mathbb{R}^q)$ is called the superalgebra of functions on $\mathbb{R}^{p|q}$. Any function $f = f(x,\xi) \in C^\infty(\mathbb{R}^{p|q})$ has a unique component expansion

$$f = f(x,\xi) = f_0(x) + \sum_i f_1^{(i)}(x)\xi_i + \sum_{ij} f_2^{(ij)}(x)\xi_i\xi_j + \cdots$$

Supermanifolds are defined as objects obtained by pasting together open pieces of $\mathbb{R}^{p|q}$. A *supermanifold* $\mathcal{M}$ is a topological space M with a sheaf of supercommutative algebras $\mathcal{F} = \mathcal{F}_\mathcal{M}$ locally isomorphic to $(\mathbb{R}^{p|q}, C^\infty(\mathbb{R}^{p|q}))$. In other words, M can be covered by open charts U_α, such that $\mathcal{F}_{U_\alpha} \simeq C^\infty(U_\alpha) \otimes \Lambda^*(\mathbb{R}^q)$. We denote the underlying manifold M of $\mathcal{M}$ by $\mathcal{M}_{rd}$.

If a coordinate system (generators of $\mathcal{F}_{U_\alpha}$) $(x^\alpha, \xi^\alpha) = (x_1^\alpha, \ldots, x_p^\alpha, \xi_1^\alpha, \ldots, \xi_q^\alpha)$ is chosen on each chart $(U_\alpha, \mathcal{F}_{U_\alpha})$, then the structure of $\mathcal{M}$ can be defined in terms of *transition functions* $G^{\alpha\beta} = (g^{\alpha\beta}, \gamma^{\alpha\beta})$:

$$x_i^\alpha = g_i^{\alpha\beta}(x^\beta, \xi^\beta) = g_{i,0}^{\alpha\beta}(x^\beta) + \sum_{k,l} g_{i,kl}^{\alpha\beta}(x^\beta)\xi_k^\beta \xi_l^\beta + \cdots, \tag{1}$$

$$\xi_j^\alpha = \gamma_j^{\alpha\beta}(x^\beta, \xi^\beta) = \sum_k \gamma_{jk}^{\alpha\beta}(x^\beta)\xi_k + \cdots, \tag{2}$$

satisfying the standard cocycle conditions $G^{\alpha\beta} \circ G^{\beta\delta} = G^{\alpha\delta}$.

The free terms $g_{i,0}^{\alpha\beta}(x^\beta)$ of the expansion (1) give the transition functions of the underlying manifold $M = \mathcal{M}_{rd}$ in the atlas (U_α, x^α). The collection of matrices $\Gamma^{\alpha\beta} = (\gamma_{jk}^{\alpha\beta}(x^\beta))$ of the linear terms in (2) is a $GL(q)$-valued 1-cocycle on M. It defines a q-dimensional vector bundle on M, the conormal bundle $N = \mathcal{F}_{\mathcal{M},\bar{1}}/\mathcal{F}^3_{\mathcal{M},\bar{1}}$ of $\mathcal{M}_{rd}$ in $\mathcal{M}$.

Remark. Let M be a manifold of dimension p and E be a vector bundle on M of rank q. Then $M_E = (M; \Lambda^*(E))$ is a $(p|q)$-dimensional supermanifold with the transition functions:

$$x_i^\alpha = g_i^{\alpha\beta}(x^\beta, \xi^\beta) = g_{i,0}^{\alpha\beta}(x^\beta), \qquad \xi_j^\alpha = \gamma_j^{\alpha\beta}(x^\beta, \xi^\beta) = \sum \gamma_{jk}^{\alpha\beta}(x^\beta)\xi_k, \tag{3}$$

with $(M_E)_{rd} = M$ and $N = \Pi(E)$, where Π is the parity change functor. Such supermanifolds are called *split*. In the C^∞-case all supermanifolds are obtained by this construction (see [4]).

1.2. Regular and singular points of functions

Standard theorems on the local behavior of functions on manifolds remain valid in the super case. We will use two of them: the implicit function theorem and the Morse lemma.

A function f on a supermanifold $\mathcal{M}$ is called *regular* or *non-singular* at the point $m \in \mathcal{M}$ if at least one partial derivative of f does not vanish at m. The implicit function theorem gives the following local description of the homogeneous non-singular functions.

Proposition. *Let f be a function non-singular at $m \in \mathcal{M}$.*

(1) *If $\bar{f} = \bar{0}$, then there exists a local coordinate system (x, ξ) at m, such that $f(x, \xi) = c + x_1$.*

(2) *If $\bar{f} = \bar{1}$, then there exists a local coordinate system (x, ξ) at m, such that $f(x, \xi) = \xi_1$.*

1.2.1. Morse points

If f is singular at $m \in \mathcal{M}$, then its second differential (the quadratic part of the Taylor expansion) at m is a well-defined (super)symmetric bilinear form on $T_m\mathcal{M}$. If f is homogeneous, then the rank $\mathrm{rk}_m f$ of the second differential of f in m is called the *rank of the singularity*. If f has in m a singularity of a maximal possible rank, m is called a *Morse singular point* of f. Morse singularities have a nice local description.

Proposition (Morse lemma [16]). *Let f be a homogeneous function on a $p|q$-dimensional supermanifold with a Morse singularity at m.*

(1) *If $\bar{f} = \bar{0}$, then $\mathrm{rk}_m f = a|b = p\,|\,2[q/2]$ and there exists a local coordinate system (x, ξ) at m, such that $f(x, \xi) = c + x_1^2 + \cdots + x_a^2 + \xi_1\xi_2 + \cdots + \xi_{b-1}\xi_b$.*

(2) *If $\bar{f} = \bar{1}$, then* $\mathrm{rk}_m f = a|a$, *where* $a = \min(p, q)$, *and there exists a local coordinate system* (x, ξ) *at* m, *such that* $f(x, \xi) = x_1\xi_1 + \cdots + x_a\xi_a$.

If f has a singularity of a non-maximal rank, then we can split f locally into a sum of two functions: one having a Morse singularity and another with vanishing quadratic part.

Proposition (Morse lemma with parameters [16]). *If f is homogeneous and singular at m with* $\mathrm{rk}_m = a|b$, *then there exists a coordinate system* $(x_1, \ldots, x_a, y_1, \ldots, y_{p-a}, \xi_1, \ldots, \xi_b, \eta_1, \ldots, \eta_{q-b})$ *around m in which* $f(x, y, \xi, \eta) = g(x, \xi) + h(y, \eta)$, *where* $\mathrm{rk}(g) = a|b$ *and* $\mathrm{rk}(h) = 0|0$.

1.3. Vector fields and differential equations on supermanifolds

1.3.1. Vector fields

Derivations of the superalgebra of functions of a supermanifold $\mathcal{M}$ are called *vector fields* on $\mathcal{M}$. In local coordinates $x_1, \ldots, x_p, \xi_1, \ldots, \xi_q$ on $\mathcal{M}$ every vector field V can be written as a differential operator $V = \sum_{i=1}^{p} f_i(\partial/\partial x_i) + \sum_{j=1}^{q} g_j(\partial/\partial \xi_j)$, where $f_i, g_j \in C^\infty(\mathcal{M})$ and the odd derivative $\partial/\partial\xi_i$ is defined by the rule $\partial\xi_i/\partial\xi_j = \delta_{ij}$.

Vector fields on a supermanifold $\mathcal{M}$ form a locally free $\mathcal{F}_{\mathcal{M}}$-module Vect($\mathcal{M}$). The superspace $T_m\mathcal{M}$ of its 0-jets in $m \in \mathcal{M}$ is called *the tangent space* to $\mathcal{M}$ at m.

The space Vect($\mathcal{M}$) is a Lie superalgebra with respect to *Poisson bracket*: $[V, U] = VU - (-1)^{\bar{V}\bar{U}}UV$ or, in local coordinates $u = (x_i, \xi_j)$,

$$\left[\sum_{i=1}^{p+q} f_i \frac{\partial}{\partial u_i}, \sum_{i=1}^{p+q} h_i \frac{\partial}{\partial u_i}\right] = \sum_{i,j}\left(f_i \frac{\partial h_j}{\partial u_i}\frac{\partial}{\partial u_j} - (-1)^{(\bar{f}_i+\bar{u}_i)(\bar{h}_j+\bar{u}_j)} h_j \frac{\partial f_i}{\partial u_j}\frac{\partial}{\partial u_i}\right). \quad (4)$$

Any two non-vanishing vector fields at a given point on a manifold are locally equivalent: they can be transformed into each other by a local coordinate change. This is still true for even vector fields on supermanifolds, but for odd or non-homogeneous vector fields this theorem is no longer valid. The explanation is that a vector field on a supermanifold does not always commute with itself. For example, odd vector fields $v_1 = \partial/\partial\xi_1$, $v_2 = \partial/\partial\xi_1 + \xi_1\partial/\partial x_1$, $v_3 = \partial/\partial\xi_1 + \xi_1 x_1 \partial/\partial x_1$ do not vanish on $\mathbb{R}^{1|1}$, but they are not equivalent to each other since $[v_1, v_1] = 0$, $[v_2, v_2] = 2\partial/\partial x_1$, $[v_3, v_3] = 2x_1\partial/\partial x_1$. There exist, however, three local normal forms for non-vanishing homogeneous vector fields on supermanifolds.

Theorem (Shander [17]). *Let V be a homogeneous vector field on a supermanifold $\mathcal{M}$ not vanishing at $m \in \mathcal{M}$.*

(1) *If V is even, then there exists a coordinate system (x, ξ) at m, such that $V = \partial/\partial x_1$.*
(2) *If V is odd and the commutator $[V, V]$ does not vanish at m, then there exists a coordinate system (x, ξ) at m, such that $V = \partial/\partial\xi_1 + \xi_1\partial/\partial x_1$.*
(3) *If V is odd and the commutator $[V, V] = 0$ in a neighborhood of m, then there exists a coordinate system (x, ξ) at m, such that $V = \partial/\partial\xi_1$.*

1.3.2. Distributions and the Frobenius theorem

An $r|s$-dimensional *distribution* F on a supermanifold $\mathcal{M}$ is a subbundle of $\mathrm{Vect}(\mathcal{M})$ locally generated by r even and s odd linearly independent vector fields. The distribution F is called *integrable* if for every point $m \in \mathcal{M}$ there exists a chart $U \ni m$ with coordinates (x, ξ), such that $F(U)$ is generated by the vector fields $\partial/\partial x_1, \dots, \partial/\partial x_r, \partial/\partial \xi_1, \dots, \partial/\partial \xi_s$.

Theorem ([18]). *A distribution F on a supermanifold $\mathcal{M}$ is integrable if and only if the space of sections of F is closed under commutator of vector fields.*

Distributions of rank $0|1$ *and homological vector fields.* In the case of a distribution generated by one vector field V the integrability condition is $[V, V] = fV$, $f \in C^\infty(\mathcal{M})$. It is automatically satisfied for distributions of rank $1|0$, since every even vector field commutes with itself. For odd vector fields $[V, V] = 2V^2$, so this condition becomes non-trivial.

An odd vector field V for which $[V, V] = 2V^2 = 0$ is called *homological*. Distributions of rank $0|1$ spanned by homological vector fields are integrable. And vice versa, any integrable $0|1$-dimensional distribution can be generated locally by a homological vector field.

1.3.3. Ordinary differential equations on supermanifolds

The three types of rectifiable vector fields on supermanifolds define three types of ordinary differential equations. A solution of the differential equation determined by vector field V is a morphism

$$F : T \times \mathcal{M} \to \mathcal{M}, \quad \text{such that } F_*(\partial_T) = V \tag{5}$$

and $\pi_T \times F : T \times \mathcal{M} \to T \times \mathcal{M}$ is a (local) diffeomorphism. Here T (time) is an interval in $\mathbb{R}$ and $\partial_T = \partial/\partial t$ if V is even, and $T = \mathbb{R}^{0|1}$, $\partial_T = \partial/\partial\theta$ if V is odd; π_T denotes the canonical projection onto T. The main difference with the purely even case is that not all differential equations with odd time θ can be solved. For example, the system $\partial x/\partial\theta = \xi$; $\partial\xi/\partial\theta = 1$ has no solutions, since it implies $\partial^2 x/(\partial\theta)^2 = 1$, whereas it should be 0 because $(\partial/\partial\theta)^2 = 0$. The straightening theorem gives the following versions of the classical existence and uniqueness theorem.

Theorem ([17]).
(1) *If V is even, then Eq.(5) has a unique solution.*
(2) *If V is odd, then (5) has a solution if and only if V is a homological vector field.*

2. Differential forms as functions

Here we consider an important example of a supermanifold. Differential forms on a manifold M become functions on a supermanifold, and the De Rham differential becomes a vector field.

2.1. Supermanifold $\widehat{M}$

Let M be a manifold of dimension n. Consider the algebra Ω^*_M of differential forms on M. In a local coordinate system $x_1, \dots, x_n$ on M the algebra Ω^*_M is isomorphic to the algebra $C^\infty(x_1, \dots, x_n) \otimes \Lambda^*(\xi_1, \dots, \xi_n)$, where $\xi_i = \mathrm{d}x_i$.

Therefore, elements of Ω^*_M become functions on the $n|n$-dimensional supermanifold $\widehat{M} = (M; \Omega^*_M)$, with the transition functions

$$x_i^\alpha = g_i^{\alpha\beta}(x^\beta), \qquad \xi_i^\alpha = \sum_j \frac{\partial g_i^{\alpha\beta}}{\partial x_j^\beta} \xi_j^\beta. \tag{6}$$

Here $g^{\alpha\beta}(x)$ are the transition functions of the manifold M. Functions on $\widehat{M}$ are, by definition, differential forms on M. This allows us to treat differential forms by techniques developed for functions. Consider, for example, the function $\phi = \xi_1 + x_1\xi_2 \in C^\infty(\widehat{M})$. It is non-singular at the origin since it has a non-zero linear term ξ_1, and by the (super analog of the) implicit function theorem it can be reduced to $\phi_0 = \xi_1$ in another coordinate system. But this may seem strange, because ϕ is just the differential form $\mathrm{d}x_1 + x_1\,\mathrm{d}x_2$, which cannot be reduced to $\phi_0 = \mathrm{d}x_1$, since ϕ_0 is closed, and ϕ is not.

The explanation of this "contradiction" is that the first reduction is performed by a coordinate transformation on $\widehat{M}$, that does not correspond to a diffeomorphism of M. In other words, the group of all diffeomorphisms of $\widehat{M}$ is much larger than the subgroup of diffeomorphisms induced by diffeomorphisms of M. Any diffeomorphism g of M induces a diffeomorphism $\hat{g} = (g, g^*)$ of $\widehat{M}$, where g^* is the pull-back map of differential forms on M.

In a local coordinate system (x_i) on M we have the following expression for $\hat{g}$:

$$x_i \mapsto g_{i,0}(x), \qquad \xi_j \mapsto \sum_k \frac{\partial g_{i,0}(x)}{\partial x_k} \xi_k, \tag{7}$$

where $\xi_i = \mathrm{d}x_i$ and the diffeomorphism g of M is given by $x_i \mapsto g_{i,0}(x)$.

Therefore, a diffeomorphism of $\widehat{M}$ is induced by a diffeomorphism of M if and only if

$$g_{i,2} = g_{i,3} = g_{i,4} = \cdots = 0 \quad \text{and} \quad g_{i,1}^k = \frac{\partial g_{i,0}(x)}{\partial x_k}. \tag{8}$$

To be able to distinguish diffeomorphisms of $\widehat{M}$ induced by diffeomorphisms of M we have to restrict the freedom on $\widehat{M}$. We can do that using an extra structure that $\widehat{M}$ possesses.

2.2. The De Rham flow

Let us recall that the algebra of differential forms on M has one more essential element—the De Rham differential d. It is an odd derivation of the $\mathbb{Z}_2$-graded algebra Ω^*_M—the ring of functions on $\widehat{M}$. This means that from the point of view of $\widehat{M}$, the De Rham differential is just an odd vector field D. It has the following expression in local coordinates (x, ξ), $\xi_i = \mathrm{d}x_i$:

$$D = \xi_1 \frac{\partial}{\partial x_1} + \xi_2 \frac{\partial}{\partial x_2} + \cdots + \xi_n \frac{\partial}{\partial x_n}. \tag{9}$$

From this expression and from formula (4), we immediately get the most important property of the De Rham differential—its homologicity. Indeed,

$$D^2 = \frac{1}{2}[D, D] = \sum_{i,j} \xi_i \frac{\partial \xi_j}{\partial x_i} = 0.$$

The De Rham vector field D on $\widehat{M}$ is homological, therefore, it can be integrated to an odd flow on the supermanifold $\widehat{M}$. This flow is a canonical action of the super Lie group $\mathbb{R}^{0|1}$ on $\widehat{M}$. Diffeomorphisms of $\widehat{M}$ which come up from M commute with this action or, in another language, are $\mathbb{R}^{0|1}$-equivariant.

Remarks.

(1) If we have a manifold X with an action of a group G on it, it would be natural to consider the algebra of invariant functions. It is isomorphic to the algebra of functions on the quotient X/G when the quotient exists. But even when there is no good quotient, the algebra of G-invariant functions X still carries important information about the geometry of the action. In our case the action is "bad": all points of the submanifold $M \subset \widehat{M}$ are invariant under the action of the De Rham flow and there is no good quotient $\widehat{M}/\mathbb{R}^{0|1}$. On the other hand, $\mathbb{R}^{0|1}$-invariant functions on the supermanifold $\widehat{M}$ coincide with closed differential forms on M. This explains why in some situations it is more natural to study closed differential forms than arbitrary ones.

(2) *$\widehat{M}$ as a super loop space*: There is another way to construct the supermanifold $\widehat{M}$, and it makes the existence of a natural $\mathbb{R}^{0|1}$-action on $\widehat{M}$ more transparent.

A remarkable feature of the geometry of supermanifolds is that the space of maps from $\mathbb{R}^{0|q}$ to an arbitrary supermanifold has a natural structure of a *finite-dimensional* supermanifold. In particular, it is easy to see that $\mathrm{Mor}(\mathbb{R}^{0|1}, M)$ is isomorphic to the supermanifold $\widehat{M}$. Since $\mathbb{R}^{0|1}$ is the only connected $0|1$-dimensional supermanifold, $\widehat{M}$ can be considered as an odd analog of the space $LM = \mathrm{Map}(S^1, M)$ of loops in M. The group S^1 acts on LM by left translations. The analogous action of the supergroup $\mathbb{R}^{0|1}$ on $\mathrm{Mor}(\mathbb{R}^{0|1}, M)$ becomes the De Rham action on $\widehat{M}$ after the identification $\mathrm{Mor}(\mathbb{R}^{0|1}, M) = \widehat{M}$.

The space LM inherits from S^1 a natural action of the infinite-dimensional group $\mathrm{Diff}\, S^1$. Similarly on $\widehat{M}$, we obtain a canonical action of the *finite-dimensional* supergroup $\mathrm{Diff}(\mathbb{R}^{0|1})$.

(3) Bernstein and Leites [6] used $\widehat{M}$ for their theory of integration of differential forms on supermanifolds. Later Witten used it in his supersymmetric interpretation of Morse theory [21].

2.3. *Equivariant diffeomorphisms of $\widehat{M}$*

Let us describe the group of diffeomorphisms of $\widehat{M}$ equivariant with respect to the canonical $\mathbb{R}^{0|1}$-action on $\widehat{M}$.

On the infinitesimal level we have to find the Lie algebra of equivariant even vector fields on $\widehat{M}$, that is, vector fields commuting with the De Rham vector field D, the generator of the action.

We will show that the Lie superalgebra of all vector fields on $\widehat{M}$ commuting with D coincides with the range of the operator

$$\mathrm{ad}_D : Y \mapsto [D, Y]. \tag{10}$$

Lemma 2.3.1. $(ad_D)^2 = 0$.

Proof. Using Jacobi identity and properties of D we have

$$(\mathrm{ad}_D)^2(Y) = [D,[D,Y]] = [[D,D],Y] - [D,[D,Y]] = -[D,[D,Y]]$$

and, therefore, $2[D,[D,Y]] = 0$. □

Proposition 2.3.2. *A vector field X on $\widehat{M}$ is D-equivariant if and only if*

$$X = [D, Y] \quad \textit{for some } Y \in \mathrm{Vect}(\widehat{M}).$$

Proof. The part *if* of the proposition is precisely the statement of the previous lemma.

To prove the *only if* part we first consider the case where M is a contractible domain in $\mathbb{R}^n$.

Let $x_1, \dots, x_n$ be coordinates on M, and $\xi_i = \mathrm{d}x_i$ the corresponding odd coordinates on $\widehat{M}$. Consider an equivariant vector field X on $\widehat{M}$ and compute the commutator of vector fields

$$X = \sum_i (a_i(x,\xi)\partial/\partial x_i + b_i(x,\xi)\partial/\partial\xi_i) \quad \text{and} \quad D = \sum \xi_i \partial/\partial x_i.$$

We have

$$\begin{aligned}[D,X] &= \sum_{ij}\left(\xi_j\frac{\partial a_i}{\partial x_j} + \xi_j\frac{\partial b_i}{\partial x_j}\right) - \sum_i (-1)^{\bar{b}_i}\frac{\partial}{\partial x_i} \\ &= \sum_i \left(D(a_i)\frac{\partial}{\partial x_i} + D(b_i)\frac{\partial}{\partial \xi_i} - (-1)^{\bar{b}_i} b_i \frac{\partial}{\partial x_i}\right).\end{aligned}$$

Therefore, $[D, X] = 0$ is equivalent to

$$b_i = (-1)^{\bar{b}_i} D(a_i) \quad \text{and} \quad D(b_i) = 0.$$

Since $D^2 = 0$, the second equation follows from the first one. Therefore, we have only one condition $b_i = (-1)^{\bar{b}_i} D(a_i)$ which is equivalent to

$$X = [D, Y], \quad \text{where } Y = \sum a_i \frac{\partial}{\partial \xi_i}.$$

Now let M be an arbitrary manifold. Choose a covering $\mathcal{U}$ of M by contractible open subsets U_i, such that for any $\alpha = \{i_1, \dots, i_k\}$ the intersection $U_\alpha = \bigcap_{s=1}^k U_{i_s}$ is contractible (or empty).

Let $\mathcal{C} = \mathcal{C}(\mathcal{U}, \mathcal{V})$ be the Čech complex for the sheaf $\mathcal{V}$ of vector fields on $\widehat{M}$ and the covering $\mathcal{U}$. The operator ad_D acts on $\mathcal{V}$ as a differential which commutes with the Čech differential δ. Thus, $\mathcal{C}$ becomes a double complex whose total cohomology is isomorphic to the homology of the operator ad_D in the space $\mathrm{Vect}(\widehat{M})$. Since ad_D has trivial homology on any U_α, a standard homological argument shows that the homology of ad_D on $\mathrm{Vect}(\widehat{M})$ is also trivial. Therefore, for $X \in \mathrm{Ker}(\mathrm{ad}_D)$ we conclude that $X \in \mathrm{Im}(\mathrm{ad}_D)$. □

Corollary 2.3.3. *The component of unity of the group of equivariant diffeomorphisms of $\widehat{M}$ consists of diffeomorphisms that have the following form in any local coordinate system (x, ξ), $\xi_i = \mathrm{d}x_i$*

$$x_i \mapsto g_i(x, \xi), \qquad \xi_j \mapsto \sum_k \frac{\partial g_i(x, \xi)}{\partial x_k} \xi_k, \tag{11}$$

where

$$g_i(x, \xi) = g_{i,0}(x) + \sum_{k_1, k_2} g_{i,2}^{k_1 k_2}(x) \xi_{k_1} \xi_{k_2} + \sum_{k_1, k_2, k_3, k_4} g_{i,4}^{k_1 k_2 k_3 k_4}(x) \xi_{k_1} \xi_{k_2} \xi_{k_3} \xi_{k_4} + \cdots \tag{12}$$

are arbitrary even functions for which $x_i \mapsto g_{i,0}(x)$ is a diffeomorphism of M.

Proof. The statement follows from the fact that any even equivariant vector field X on $\widehat{M}$ has the following coordinate form:

$$X = \sum_i \left(a_i(x, \xi) \frac{\partial}{\partial x_i} + D(a_i) \frac{\partial}{\partial \xi_i} \right),$$

where a_i are arbitrary even functions on $\widehat{M}$. □

3. Equivariant Morse lemma for the De Rham flow

In this section we will give a proof of the equivariant Morse lemma for the De Rham flow.

Arnold in [2] considered an action of a compact Lie group G on $\mathbb{R}^n$ with a fixed point at 0. He proved that every G-invariant function $f \in C^\infty(\mathbb{R}^n)$ which has at 0 a non-degenerate critical point with critical value 0 can be reduced to its quadratic part by an equivariant change of variables.

This theorem can be generalized for functions with degenerate critical points. Namely, if f has at 0 a critical point of rank r (the rank of a critical point is the rank of the second differential $\mathrm{d}^2 f$ at this point), then there exists an equivariant diffeomorphism reducing f to a function $f(x_1, \ldots, x_n) = f_2(x_1, \ldots, x_r) + \tilde{f}(x_{r+1}, \ldots, x_n)$, where f_2 is a non-degenerate quadratic form and the quadratic part of $\tilde{f}$ at 0 vanishes.

In this section we prove an analog of the equivariant Morse lemma for the action of the supergroup $\mathbb{R}^{0|1}$ on $\widehat{M}$ generated by the De Rham vector field D. This theorem generalizes both the usual Morse lemma and the Darboux theorem. Since all our considerations are local, we put $M = \mathbb{R}^n$ and, consequently, $\widehat{M} = \mathbb{R}^{n|n}$.

Theorem 3.1. *Let f be a homogeneous (even or odd) function on $\mathbb{R}^{n|n} = \widehat{\mathbb{R}^n}$ invariant with respect to the De Rham flow. If f vanishes at 0 and has a singular point of the maximal possible rank, then it can be reduced to its quadratic part f_2 at 0 by a local equivariant change of coordinates.*

We begin with considering quadratic parts of D-invariant functions.

Lemma 3.2. *Let f be a D-invariant homogeneous function on $\widehat{M}$ with a singularity at the point $p \in M$. Let $\dim M = n$. Then*

$$\mathrm{rk}_p(f) = 0|2r, \quad \text{where } 2r \le n \quad \text{if } \bar{f} = \bar{0},$$
$$\mathrm{rk}_p(f) = r|r, \quad \text{where } r \le n \quad \text{if } \bar{f} = \bar{1}.$$

Proof. Consider the quadratic part f_2 of f in a local coordinate system $(x_i, \xi_i), \xi_i = Dx_i$ around p. Invariant functions on $\widehat{M}$ are closed differential forms on M. Therefore, when f is even, f_2 cannot depend on x_i, so $f_2 = \sum a_{ij}\xi_i\xi_j$ and by a linear transformation of ξ_i we can take this skew symmetric bilinear form to the canonical form

$$f_2 = \xi_1\xi_2 + \xi_3\xi_4 + \cdots + \xi_{2r-1}\xi_{2r}, \tag{13}$$

where $2r < n$.

In the case when f is odd $f_2 = \sum_{i,j} b_{ij}x_i\xi_j$, and $Df_2 = 0$ if and only if $f_2 = Dg$ for $g = \sum_{i,j} c_{ij}x_ix_j$. Diagonalizing the quadratic form g by a linear change of coordinates x_i we get

$$f_2 = D\left(\frac{1}{2}\sum_{i=1}^{r} \pm x_i^2\right) = x_1\xi_1 \pm x_2\xi_2 \pm \cdots \pm x_r\xi_r. \qquad \square \tag{14}$$

To prove Theorem 3.1 we have to consider three cases:

(i) f is odd;

(ii) f is even, and $n = \dim M$ is even;

(iii) f is even, and n is odd.

Let us first show that (iii) follows from (ii).

Reduction of the part (iii) *to* (ii). Since the rank of f is maximal, the dimension of $(\mathrm{Ker}\, f_2)_{\bar{1}} \subset T_{\bar{1}}\widehat{M}$ is equal to 1 at all points. Let V be a non-vanishing odd section of $\mathrm{Ker}\, f_2$ considered as a line bundle over M. Then V is a homological vector field, such that $Vf = 0$ and $X = [D, V] \neq 0$ at all points. Therefore, the vector fields X and V span a regular $1|1$-dimensional distribution S on $\widehat{M}$. Let us show that this distribution is integrable and D-invariant.

(1) *Frobenius condition*

$$[V,V]=0 \quad \text{and} \quad [V,X]=[V,[D,V]]=[[V,D],V]-[D,[V,V]]$$
$$=[[V,D],V]=-[V,[V,D]]=-[V,[D,V]]$$

from where, by Lemma 2.3.1, $[V,[D,V]]=0$. This establishes the integrability of S.

(2) *D-invariantness*

$$[D,V]=X \quad \text{and} \quad [D,X]=[D,[D,V]]=[[D,D],V]-[D,[D,V]]$$
$$=-[D,[D,V]]=0 \quad \text{by 2.3.1.}$$

In addition, $Xf=0$, since $Vf=Df=0$. Therefore, there exists a local D-equivariant submersion $p:\widehat{M}\to\widehat{N}$, such that $\operatorname{Ker}(p_*)=S$ and f is constant along the fibers of p. This means that f pushes down to a function $\bar{f}$ on $\widehat{N}$ which is still D-invariant and has maximal rank $2r$.

But $\dim N=(n-1|n-1)=(2r|2r)$, therefore we can apply the case (ii) of the theorem and find a D-equivariant coordinate change on $\widehat{N}$ reducing $\bar{f}$ to $\xi_1\xi_2+\cdots+\xi_{2r-1}\xi_{2r}$. Now, if we augment the new coordinates (x,ξ) on $\widehat{N}$ by equivariant coordinates (y,η) along the fibers of p, we obtain an equivariant coordinate transformation on $\widehat{M}$ taking f to f_2.

Proof of (i) *and* (ii). After a linear coordinate change, we may assume that $f=f_2+\tilde{f}$, where f_2 is the quadratic part of f in one of the canonical forms (13) or (14) and $\tilde{f}$ has at least order 3 at 0 (i.e. $\operatorname{rk}(\tilde{f})=0|0$). The rank of f is maximal if $r=[n/2]$ in the case of even f, and $r=n$ if f is odd.

We are going to use the homotopy method of Moser [13]. Namely, instead of looking for a single (local) diffeomorphism g of $\widehat{M}$, such that $g^*(f)=f_2$, we will consider the family of functions

$$F_t=f_2+t\tilde{f}$$

and find a one-parameter family of equivariant diffeomorphisms g_t, $t\in[0,1]$, such that

$$g_t^*(F_t)=f_2. \tag{15}$$

Differentiating (15) by t, we get

$$0=\frac{\mathrm{d}}{\mathrm{d}t}F_t(g_t(x,\xi))=\tilde{f}(g_t(x,\xi))+X_t(F_t(g_t(x,\xi))), \tag{16}$$

where

$$X_t=\frac{\mathrm{d}}{\mathrm{d}t}g_t(x,\xi).$$

Since g_t is a diffeomorphism, Eq. (16) for fixed t is equivalent to

$$X_tF_t(y,\eta)=-\tilde{f}(y,\eta) \tag{17}$$

in a new coordinate system $(y,\eta)=g_t(x,\xi)$.

If we find a smooth family of even vector fields X_t in a neighborhood of $(0, 0)$ satisfying (17) and the condition $X_t(0, 0) = 0$ for all $t \in [0, 1]$, then it will give us a required family g_t. Indeed, the existence theorem for ODE on supermanifolds gives g_t only locally on t, but the condition $X_t(0, 0) = 0$ guarantees that it will extend to the whole segment $[0, 1]$. The resulting family will be defined in a smaller neighborhood of the origin than the vector field X, but this is all we need. The family of diffeomorphisms will be equivariant with respect to the De Rham flow if its generator X_t commutes with the De Rham vector field D.

Existence of such vector field is given by the following lemma.

Lemma 3.3. *In the situation of Theorem* 3.1, *let h be a germ of a D-invariant function on $\widehat{M}$ vanishing at* 0, *such that $\bar{h} = \bar{f}$ and* $\mathrm{rk}(h) = 0|0$. *Then there exists (a germ of) an equivariant even vector field X, such that $Xf = h$ and $X(0) = 0$.*

Proof. To finish the proof of Theorem 3.1, we need to consider only the cases (i) and (ii). The statement of the lemma in the case (iii) will follow from the theorem.

Denote by $\mathbf{m} \subset C^\infty(\mathbb{R}^{n|n})$ the ideal of functions vanishing at the origin. The De Rham vector field D is homogeneous of degree 0 with respect to $\mathbf{m}$. This means that $D(\mathbf{m}^k) \subset \mathbf{m}^k$.

Since $f \in \mathbf{m}^2, h \in \mathbf{m}^3$ and $Df = Dh = 0$, we know from the Poincaré lemma that there exist $u, w \in C^\infty(\mathbb{R}^{n|n})$, such that

$$f = Du, \quad h = Dw, \qquad u \in \mathbf{m}^2, \ w \in \mathbf{m}^3.$$

From the description of D-equivariant vector fields in 2.3.2, it follows that the vector field X should be of the form $X = [D, Y]$ for some $Y \in \mathrm{Vect}_{\bar{1}}(\mathbb{R}^{n|n})$. The equation $Xf = h$ can now be rewritten as

$$[D, Y]f = Dw \quad \text{or} \quad DYf = Dw.$$

Therefore, it is enough to find an odd vector field

$$Y = \sum \alpha_i(x, \xi)\frac{\partial}{\partial x_i} + \sum a_i(x, \xi)\frac{\partial}{\partial \xi_i}, \tag{18}$$

such that

$$Yf = w \quad \text{and} \quad Y(0) = 0. \tag{19}$$

Now we consider the cases (i) and (ii) separately.

(i) If $\bar{f} = \bar{1}$, then after a linear change of coordinates on $\mathbb{R}^{n|n}$, we can write

$$f = \sum x_i \xi_i + \tilde{f}, \quad \text{where } \tilde{f} \in \mathbf{m}^3.$$

Therefore, the functions

$$y_i = \frac{\partial f}{\partial \xi_i}, \qquad \eta_i = \frac{\partial f}{\partial x_i}$$

provide a new coordinate system in a neighborhood of the origin.

Since $w \in (\mathbf{m}^3)_{\bar{1}}$, there exist $a_i \in (\mathbf{m}^2)_{\bar{0}}$, $\alpha_i \in (\mathbf{m}^2)_{\bar{1}}$, such that $w = \sum a_i \eta_i + \sum \alpha_i y_i$. Then the vector field

$$Y = \sum a_i \frac{\partial}{\partial \xi_i} + \sum \alpha_i \frac{\partial}{\partial x_i}$$

satisfies Eq. (19).

(ii) If $\bar{f} = \bar{0}$ and n is even, then the maximal rank condition implies that

$$f = \xi_1 \xi_2 + \cdots + \xi_{n-1} \xi_n + \tilde{f}, \quad \text{where } \tilde{f} \in (\mathbf{m}^3)_{\bar{0}},$$

so the n odd functions $\eta_i = \partial f / \partial \xi_i$ are independent at 0 and we can find n even functions y_i, such that together they give a local coordinate system around $0 \in \mathbb{R}^{n|n}$. Since $w \in (\mathbf{m}^3)_{\bar{1}}$, we can write w as $w = \sum a_i \eta_i$, and the vector field

$$Y = \sum a_i \frac{\partial}{\partial \eta_i}$$

gives a solution to (19). □

Corollary 3.4.

(i) *Let ω be a closed 2-form in $\mathbb{R}^n$ of maximal rank $2[n/2]$ at 0. Then it can be reduced to a canonical Darboux form*

$$\omega_0 = \mathrm{d}x_1 \wedge \mathrm{d}x_2 + \cdots + \mathrm{d}x_{2r-1} \wedge \mathrm{d}x_{2r}. \tag{20}$$

(ii) *Let $\alpha = \sum f_i(x)\,\mathrm{d}x_i$ be a closed 1-form in $\mathbb{R}^n$, such that $f_i(0) = 0$ and the differentials $\mathrm{d}f_i(0)$ are linearly independent. Then α can be reduced to a canonical form*

$$\alpha_0 = x_1\,\mathrm{d}x_1 + \cdots + x_n\,\mathrm{d}x_n. \tag{21}$$

Proof. Theorem 3.1 gives equivariant diffeomorphisms of $\widehat{\mathbb{R}}^n$ which reduce forms ω and α to (20) and (21). These diffeomorphisms have coordinate expressions (11), but, since ω and α are homogeneous on ξ_i with respect to $\mathbb{Z}$-grading in $\Omega^*(\mathbb{R}^n)$, the components $g_{i,0}(x)$ of (12) give diffeomorphisms of $\mathbb{R}^n$ which perform the reductions.

Indeed, let $x \mapsto g(x, \xi)$ be the transformation $g_0(x) = g(x, 0)$. Then g_0 is a diffeomorphism of M. If the form ω is homogeneous (i.e. $\omega = \omega_2 \in \Omega^2_M$), then $\tilde{\omega} = \omega_2(x) + \cdots$ and the quadratic part of $\tilde{\omega}$ is determined by g_0. Therefore, g_0 is a required diffeomorphism of M. □

Concluding remarks

(1) The part (ii) of the corollary is, in fact, equivalent to the standard Morse lemma. Indeed, the map $f \mapsto \alpha = \mathrm{d}f$ is a one-to-one correspondence between functions on $\mathbb{R}^n$ with a non-degenerate singularity at 0 of value 0 and the closed 1-forms described in Corollary 3.4(ii).

(2) The equivariant Morse lemma gives more than just the Darboux theorem and the Morse lemma. It states, in particular, that a closed non-homogeneous differential form

$\omega = \omega_1 + \omega_3 + \cdots, \omega_k \in \Omega_M^k$ with non-degenerate $\omega_1 = \sum a_i(x)\,\mathrm{d}x_i, a_i(0) = 0$ can be taken to the form $\sum \pm x_i\,\mathrm{d}x_i$ by a transformation

$$x \mapsto g_0^0(x) + g_2^{ij}(x)\,\mathrm{d}x_i \wedge \mathrm{d}x_j + \cdots,$$

and a closed even form $\omega = \omega_2 + \omega_4 + \cdots$, with a non-degenerate ω_2 can be reduced to a Darboux canonical form.

(3) The original proof of the equivariant Morse lemma given by Arnold [2] cannot be generalized to the super case since Arnold used invariant integration, and it does not exist for $\mathbb{R}^{0|1}$-actions.

Moreover, as we will show somewhere else, the equivariant Morse lemma is not valid for a general $\mathbb{R}^{0|1}$-action. It is correct, though, for the large class of $\mathbb{R}^{0|1}$-actions of generic type [19], and, in particular, it generalizes the Darboux theorem for supermanifolds.

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